STRESS CONCENTRATION IN AN ANISOTROPIC BODY CONTAINING A PERIODIC SYSTEM OF THIN ELASTIC INCLUSIONS

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This article analyzes stress concentration in an anisotropic body containing a periodic system of thin elastic inclusions. Using the plane problem of elasticity and the Winkler model, the author optimizes the inclusion shape to achieve a uniform internal stress state. An exact analytical solution is derived for this optimal shape, along with formulas for the resulting stresses and the stress intensity factor.

Keywords: anisotropic body, elastic inclusions, shape optimization, stress intensity factor

Basic equations of the plane problem of elasticity theory for an anisotropic body. For the generalized plane stress state of an anisotropic body, the strains depend on the stresses according to the following relations:

$$\begin{aligned}\varepsilon_{xx} &= a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{16}\tau_{xy},\\ \varepsilon_{yy} &= a_{12}\sigma_{xx} + a_{22}\sigma_{yy} + a_{26}\tau_{xy},\\ 2\varepsilon_{xy} &= a_{16}\sigma_{xx} + a_{26}\sigma_{yy} + a_{66}\tau_{xy}.\end{aligned} \tag{1}$$

Here a_{ij}(i, j = 1, 2 … 6) are the elastic constants.

It should be noted that under conditions of plane strain, the relationship between the components of the strain and stress tensors has the same form, but the constants a_{ij} should be replaced with constants b_{ij}, where:

$$b_{11} = \frac{a_{11}a_{33} - a_{13}^2}{a_{33}}, \qquad b_{12} = b_{21} = \frac{a_{12}a_{33} - a_{13}a_{23}}{a_{33}},$$

$$b_{22} = \frac{a_{22}a_{33} - a_{23}^2}{a_{33}}, \qquad b_{16} = b_{61} = \frac{a_{16}a_{33} - a_{13}a_{36}}{a_{33}},$$

$$b_{66} = \frac{a_{66}a_{33} - a_{36}^2}{a_{33}}, \qquad b_{26} = b_{62} = \frac{a_{26}a_{33} - a_{23}a_{36}}{a_{33}}.$$

The equilibrium equations, in the absence of body forces, will be satisfied if the stress function F satisfies the differential equation:

$$a_{22}\frac{\partial^4 F}{\partial x^4} - 2a_{26}\frac{\partial^4 F}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 F}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 F}{\partial x \partial y^3} + a_{11}\frac{\partial^4 F}{\partial y^4} = 0 \tag{2}$$

The general expression for the function F depends on the parameters μ_k – the roots of the algebraic characteristic equation:

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0. \quad (3)$$

When the roots μ_k are unequal, the general expression for the function F can be represented through two analytic functions χ_1, χ_2 of the complex variables $z_k = x + \mu_k y$ (k=1, 2):

$$F(x, y) = Re[\chi_1(z_1) + \chi_2(z_2)]. \quad (4)$$

The stresses and displacements can be expressed in terms of these analytic functions by the following relations:

$$\begin{aligned} \sigma_{xx} &= 2\,Re\{\mu_1^2\Phi_1(z_1) + \mu_2^2\Phi_2(z_2)\}, \\ \sigma_{yy} &= 2\,Re\{\Phi_1(z_1) + \Phi_2(z_2)\}, \\ u_x(z) &= 2\,Re[p_1\varphi_1(z_1) + p_2\varphi_2(z_2)], \\ u_y(z) &= 2\,Re[q_1\varphi_1(z_1) + q_2\varphi_2(z_2)]. \end{aligned} \quad (5)$$

Here $p_k = a_{11}\mu_k^2 + a_{12} - a_{16}\mu_k$, $q_k = a_{12}\mu_k + \frac{a_{22}}{\mu_k} - a_{26}$, $\Phi_1(z_1) = \varphi_1'(z_1) = \chi_1''(z_1)$, $\Phi_2(z_2) = \varphi_2'(z_2) = \chi_2''(z_2)$.

Formulation of the problem. We consider an infinite anisotropic body containing a periodic system of identical collinear thin elastic inclusions with the same mechanical and geometric parameters (Fig. 1). Conditions of perfect contact are satisfied at the material interfaces. We associate a system of local rectangular coordinates x_k, y_k with each inclusion such that the x_k-axes coincide with the axis of inclusion placement. An external tensile load of intensity σ^∞ is applied to the body at infinity.

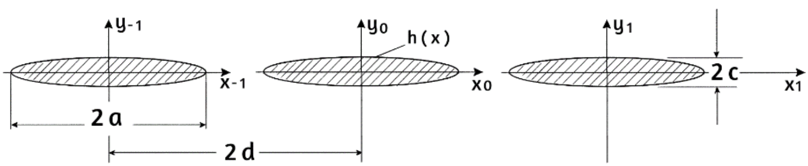

Fig. 1. Periodic system of thin elastic inclusions

To solve the problem, we utilize the Winkler model for a thin elastic inclusion. According to this model, the normal stresses σ_{yy}^* within the inclusions are expressed as:

$$\sigma_{yy}^* = \frac{[u_y^*]}{2h(x)}E_1, \quad (6)$$

$$u_y^* = u_y + u_y^0,\ u_y^0 = \sigma^\infty a_{22}h(x)$$

The square brackets denote the jump in the function across the plane $y = 0$. Here, $2h(x)$ is the thickness of the inclusion, and E_1 is the Young's modulus of the inclusion material.

The problem of a body with inclusions is reduced, based on the principle of superposition, to the sum of two stress states: (a) a body with thin cylindrical cavities, to the surfaces of which forces are applied:

$$\sigma_{yy} = \sigma_{yy}^{*} - \sigma^{\infty};\ \sigma_{xy} = 0 \tag{7}$$

and (b) a homogeneous body loaded by external forces σ_{∞}. The solution to the latter problem is trivial and well-known.

Integral equation and its solution. Solving problem (a), considering that all cavities are in identical conditions, leads to the integral equation:

$$\int_{-a}^{a} [u_y'] \operatorname{ctg}\frac{\pi(t-x)}{2d}\,dt \quad = \frac{4d}{\operatorname{Re}\left(\frac{1}{\pi i}\frac{\mu_1-\mu_1}{q_1\mu_2-q_2\mu_1}\right)}\left(\frac{[u_y]}{2h(x)}E_1 + (E_1\alpha_{22}-1)\sigma^{\infty}\right) \tag{8}$$

In the general case of the inclusion shape $h(x)$, the solution to the integral equation can be obtained by numerical methods.

We set the task of finding an inclusion shape for which the stress state within them would be homogeneous. Such optimization of the shape of reinforcing elements can arise in multicomponent systems operating under specific service conditions. For this, the following condition must be met:

$$\frac{[u_y^{*}]}{2h(x)} = \text{ const.} \tag{9}$$

The exact solution to the equation in this case has the form:

$$[u_y] = \frac{2\sigma_{\infty}(1-E_1\alpha_{22})}{\pi(1+2\beta E_1\alpha_{22})Re\left(\frac{1}{\pi i}\cdot\frac{\mu_2-\mu_1}{q_1\mu_2-q_2\mu_1}\right)}\ln\left|\frac{\cos\frac{\pi x}{2d}+\sqrt{\sin^2\frac{\pi\lambda}{2}-\sin^2\frac{\pi x}{2d}}}{\cos\frac{\pi x}{2d}-\sqrt{\sin^2\frac{\pi\lambda}{2}-\sin^2\frac{\pi x}{2d}}}\right|, \tag{10}$$

$$\lambda = \frac{a}{d},\ \beta = \frac{d}{\pi c}\ln\frac{1+\sin\frac{\pi\lambda}{2}}{1-\sin\frac{\pi\lambda}{2}}.$$

The stresses in the inclusions will be uniform and are given by:

$$\sigma_{yy}^{*} = \frac{\sigma^{\infty}E_1\alpha_{22}(1+2\beta)}{(1+2\beta E_1\alpha_{22})\operatorname{Re}\left(\frac{1}{\pi i}\cdot\frac{\mu_2-\mu_1}{q_1\mu_2-q_2\mu_1}\right)}. \tag{11}$$

The shape of the inclusions corresponding to such a stress distribution is described by the equation:

$$h(x) = \frac{d}{\pi\beta} \ln \left| \frac{\cos\frac{\pi x}{2d} + \sqrt{\sin^2\frac{\pi x}{2} - \sin^2\frac{\pi x}{2d}}}{\cos\frac{\pi x}{2d} - \sqrt{\sin^2\frac{\pi x}{2} - \sin^2\frac{\pi x}{2d}}} \right| \tag{12}$$

As $d \to \infty$, the shape of the inclusions approaches an ellipse with semi-axes a and c , $(a > c)$. When $d \to a$, it degenerates to a rectangular shape with dimensions $2a \times 2c$.

The stress intensity factors K_I near the inclusions can be calculated based on the known displacements u_y using the formula:

$$K_I = \lim_{r\to 0} \frac{u_y(\mathrm{r})}{\sqrt{2r}\,\mathrm{Re}\left(\frac{\mu_2 q_1 - \mu_1 q_2}{\mu_2 - \mu_1}\right)}. \tag{13}$$

From this and the previous relations, we obtain:

$$K_I = \sigma^\infty \sqrt{\pi a}(1 - E_1 a_{22})(1 + 2E_1 a_{22}\beta)^{-1} \sqrt{\frac{2}{\pi\lambda}\mathrm{tg}\frac{\pi\lambda}{2}} \tag{14}$$

It should be noted that the stress intensity factors describe the stress distribution in a small annular zone in the vicinity of the inclusion tips. Directly near the inclusion in the matrix, the stress can be determined based on the condition of compatibility of deformations of the inclusion and the matrix at the point $x = a, y = 0$ and Hooke's law for an anisotropic body:

$$\sigma_{yy} = \frac{\sigma^\infty(1+2\beta)}{a_{22}(1+2\beta E_1 a_{22})\mathrm{Re}\left(\frac{1}{\pi i}\cdot\frac{\mu_2-\mu_1}{q_1\mu_2 - q_2\mu_1}\right)}. \tag{15}$$

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