SPECTRAL DENSITY BIAS ESTIMATION FOR PERIODICALLY NON-STATIONARY RANDOM PROCESSES

ROMAN KHMIL[1,2], ROMAN YUZEFOVYCH[1,2]

1. Karpenko Physico-mechanical Institute of NAS of Ukraine
2. Lviv Polytechnic National University, Ukraine

Correlogram-based estimation of the instantaneous spectral density of periodically non-stationary random processes is investigated for both continuous and discrete signals derived using coherent covariance methods. Expressions for the asymptotic bias and variance were obtained, first- and second-order aliasing effects are examined. Proposed models were applied to treaty of the amplitude- and phase-modulated signals, simulated and real time-series examples for obtaining spectral-density and cyclic-components estimations.

Keywords: periodically non-stationary random process; instantaneous spectral density; spectral component

Introduction. Using periodically non-stationary random process (PNRP) mean function, covariance function, the instantaneous spectral density and their Fourier coefficients allow to investigate the specific properties of stochastic recurrence for different physical phenomena. To compute these quantities on the basis of experimental data the methods for PNRP statistical covariance and spectral analysis were developed [1–6, 9, 11]. The properties of estimators for the mean, covariance function and their Fourier coefficients, which are determined by coherent (synchronous averaging) and component methods, least square method, linear comb-like and band-filtration methods, were analyzed in detail for the cases of known or unknown non-stationary period. It was obtained the simple and practically useful formulae for biases and variances of the estimators that allow to compute relatively easily the estimation mean-square errors in dependency on realization length, sampling interval, point of correlogram cut-off value and parameters which describe PNRP covariance structure. The required values of these errors are the base for grounded choice of the parameters for statistical processing of the experimental data.

The estimation of the instantaneous spectral density. To estimate the spectral density one can, use the correlogram method of Blakman-Tukey:

$$\hat{f}(\omega,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{R}(t,\tau)k(\tau)e^{-i\omega\tau}d\tau. \tag{1}$$

Here $\hat{R}(t,\tau)$ – is the covariance function estimator and $k(\tau)$ – covariance window: $k(-\tau)=k(\tau)$, $k(0)=1$, $k(\tau)=0$ for $|\tau|\geq\tau_m$. The coherent covariance function estimator can be calculated using the formula:

$$\hat{R}(t,\tau)=\frac{1}{N}\sum_{n=0}^{N-1}[\xi(t+nP)-\hat{m}(t+nP)][\xi(t+\tau+nP)-\hat{m}(t+\tau+nP)],$$

where

$$\hat{m}(t) = \frac{1}{N}\sum_{n=0}^{N-1} \xi(t + nP),$$

and N is the number of the averaged realization period. Further we will suppose that realization length is such that $T = NP + \tau_m$, where τ_m is a point of correlogram cut-off and that mean function is estimated only for $t \in [0, P]$ and $\hat{m}(t + nP) = m(t)$, when $t \in [0, P]$. Then

$$\hat{R}(t,\tau) = \frac{1}{N}\sum_{n=0}^{N-1}[\xi(t + nP) - \hat{m}(t + nP)][\xi(t + \tau + nP) - \hat{m}(t + \tau + nP)]. \quad (2)$$

If the covariance function $R(t,\tau)$ is calculated using statistics (2) the estimator (1) is asymptotically biased (for finite τ_m). Its than bias $\varepsilon\left[\hat{f}(\omega, t)\right] = Ef(\omega, t) - f(\omega, t)$ for $N \to \infty$ in the first approximation is defined by formula

$$\varepsilon_{as}\left[\hat{f}(\omega, t)\right] = \left[\frac{1}{2}\int_{\infty}^{\infty} \omega_{1z}\lambda(\omega_1)d\omega_l\right] f^{//}(\omega, t), \quad (3)$$

where

$$\lambda(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(\tau)e^{-i\omega\tau}d\tau.$$

Since the mathematical expectation of estimator (2) is equal to [11]:

$$E\hat{R}(t,\tau) = R(t,\tau) - \frac{1}{N}\sum_{n=-N+1}^{N-1}\left(1 - \frac{|n|}{N}\right)R(t, \tau + nP),$$

then

$$E\hat{f}(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(\tau)R(t,\tau)e^{-i\omega\tau}d\tau -$$
$$- \frac{1}{N}\sum_{n=N+1}^{N-1}\left(1 - \frac{|n|}{N}\right)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} k(\tau)R(t, \tau + nP)e^{-i\omega\tau}d\tau\right]. \quad (4)$$

Taking into account the representations

$$R(t,\tau) = \int_{-\infty}^{\infty} f(\omega, t)e^{i\omega\tau}d\omega, \; k(\tau) = \int_{-\infty}^{\infty} \lambda(\omega)e^{i\omega\tau}d\omega. \quad (5)$$

After substituting to (4) relations (5) we have:

$$E\hat{f}(\omega, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\omega_1, t)\lambda(\omega_2)\left[1 - \frac{\sum_{n=-N+1}^{N-1}\left(1 - \frac{|n|}{N}\right)e^{i\omega_1 nP}}{N}\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega_1 - \omega + \omega_2)u}du\right]d\omega_1 d\omega_2.$$

The sum under the integral can be re-wrote in the form:

$$\frac{1}{N}\sum_{n=-N+1}^{N-1}\left(1-\frac{|n|}{N}\right)e^{i\omega_1 nP}=\frac{1}{N^2}\sum_{m,n=0}^{N-1}e^{i\omega_1(m-n)P}.$$

Taking into consideration the equalities

$$\frac{1}{N}\sum_{m=0}^{N-1}e^{i\omega_1 mP}=e^{i\frac{\omega}{2}(N-1)\frac{P}{2}}\frac{\sin\omega N\frac{P}{2}}{N\sin\omega\frac{P}{2}},\ \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega u}du=\delta(\omega),$$

here $\delta(\omega)$ is the Dirac delta function, we have:

$$E\hat{f}(\omega,t)=\int_{-\infty}^{\infty}f(\omega_1,t)\lambda(\omega-\omega_1)[1-g(\omega_1,N)]d\omega_1,$$

where

$$g(\omega,N)=\frac{\sin^2\left(N\omega\frac{P}{2}\right)}{N^2\sin^2\left(\omega\frac{P}{2}\right)}.$$

The function $g(\omega,N)$ for $N\to\infty$ degenerates into series of unit pulse functions with period $2\pi/P$. Thus, asymptotical value of bias is equal to:

$$\varepsilon_{qs}\left[\hat{f}(\omega,t)\right]=\int_{-\infty}^{\infty}f(\omega-\omega_l,t)\lambda(\omega_l)d\omega_l-f(\omega,t), \qquad (6)$$

Parameters of smoothing covariance windows $k(\tau)$ and the point of correlogram cut-off τ_m are chosen so that spectral windows $\lambda(\omega)$ have a form of the sharp peaks at point $\omega=0$. Assume that the spectral density is the smooth frequency function we can represent its in the form of the Taylor series in the neighborhood of point ω:

$$f(\omega-\omega_1,t)=f(\omega,t)-f^{/}(\omega,t)\omega_1+f^{//}(\omega,t)\frac{\omega_1^2}{2}\ldots.$$

Taking into account this series and the equalities $\lambda(\omega)=\lambda(-\omega)$ and $\int_{-\infty}^{\infty}\lambda(\omega)d\omega=1$ for the asymptotic bias (6) at the first approximation we have formula (3).

Conclusions. It follows from expression (3) that the estimators of the smoother spectral densities have smaller bias, they estimators have a underestimated values near spectral density maximums and have overestimated values near its minimums, and the values of the bias $\varepsilon_{qs}\left[\hat{f}(\omega,t)\right]$ decreases width decreasing spectral window width.

1. *Dragan Ya., Javorskyj I. Rhythms of sea waves and underwater acoustic signals // Naukova Dumka, Kyiv. – 1980 (in Russian).*
2. *Gardner W.A. Introduction to random processes with applications to signal and systems // Macmillan, New York. – 1985.*
3. *Gardner W.A. The spectral correlation theory of cyclostationary time-series // Signal Process. – 1986. – 19. – P. 13–36. https://doi.org/10.1016/0165-1684(86)90092-7*
4. *Dragan Ya., Rozhkov V., Javorskyj I. The Methods of Probabilistic Analysis of Oceanological Rhythms //Gidrometeoizdat, Leningrad. – 1987 (in Russian).*
5. *Cyclostationarity in communications and signal processing, ed. by Gardner W.A. // IEEE Press, New York. – 1994.*
6. *Hurd H.L., Miamee A. Periodically correlated random sequences, spectral theory and practice // Wiley, New York. – 2007. https://doi.org/10.1002/9780470182833*
7. *Antoni J. Cyclostationary by examples // Mech. Syst. Signal Process. – 2009. – 23. – P.987–1036. https://doi.org/10.1016/j.ymssp.2008.10.010*
8. *Napolitano A. Generalizations of cyclostationary signal processing, spectral analysis and applications // IEEE Press, Wiley. – 2012. https://doi.org/10.1002/9781118437926*
9. *Javorskyj I., Mathematical models and analysis of stochastic oscillations // Karpenko Physico-Mechanical Inst. NASU. – Lviv. – 2013 (in Ukrainian).*
10. *Martin W., Flandrin P. Wigner-Ville spectral analysis of non-stationary processes// IEEE Transactions on Acoustics, Speech, and Signal Processing – 1985. – 33. – P.1461–1470. https://doi.org/10.1109/TASSP.1985.1164760*
11. *Napolitano A., Cyclostationary Processes and time series. Theory, applications and generalizations // Elsevier, Academic Press. – 2020. https://doi.org/10.1016/B978-0-08-102708-0.00034-0*