BENDING OF MULTI-CONNECTED PLATES WITH ELLIPTIC INCLUSIONS

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A new approximate method for determining the stress state of a thin anisotropic plate with elastic inclusions is proposed. The results of numerical studies are described, which made it possible to reveal the influence of the stiffness of elastic inclusions, the distances between them, and their geometric characteristics on the values of the bending moments arising in the plate. The issue of using moment's intensity factors was considered in case of sufficiently rigid or flexible linear inclusions.

Keywords: bending, plates, stress concentration, mathematical model, numerical analysis

Introduction. Despite the substantial practical relevance of analyzing the stress–strain state of thin plates under transverse bending, especially when foreign inclusions are involved, numerous engineering challenges remain unresolved. This is primarily due to the substantial mathematical and computational complexities that, until recently, were difficult to address.

Problem Statement and Solution Method. Consider an infinite anisotropic plate occupying a multiply connected domain S , bounded by the contours of elliptical holes L_l $\left(l=\overline{1,\mathrm{L}}\right)$. Elastic inclusions with corresponding domains $S^{(l)}$ made of different materials, free of initial stresses, are embedded in these holes (Fig. 1). The inclusions are assumed to be in perfect mechanical contact with the plate. At infinity, the plate is subjected to constant bending moments M_x^∞ , M_y^∞ , H_{xy}^∞ .

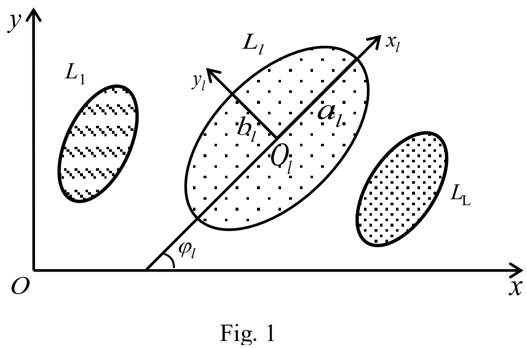

Fig. 1

To address this problem, we use the method of complex potentials from the applied theory of anisotropic plate bending [1]. The problem is thereby reduced to determining the complex potentials for both the matrix plate $W_k'(z_k)$ and each inclusion $W_k'^{(l)}(z_k^{(l)})$, which are obtained from the corresponding boundary conditions on the inclusion contours.

The complex potentials $W_k'(z_k)$ for the matrix plate are expressed as functions of generalized complex variables $z_k = x + \mu_k y$, where μ_k – the roots of the characteristic equation [2].

By applying the conformal mapping of the exterior of the unit circle $|\zeta_{kl}| \geq 1$ onto the exterior of the ellipses L_{kl}, the complex potentials can be represented as

$$W_k'(z_k) = \Gamma_k z_k + \sum_{l=1}^{L} \sum_{p=1}^{\infty} \frac{a_{klp}}{\zeta_{kl}^p},$$

where Γ_k are obtained from a system of equations that depends on μ_k, the material deformation coefficients, and the bending moments at infinity; a_{klp} – unknown quantities; and ζ_{kl} are variables defined by the relations

$$z_k = z_{kl} + R_{kl}\left(\zeta_{kl} + \frac{m_{kl}}{\zeta_{kl}}\right);$$

$$z_{kl} = x_{0l} + \mu_k y_{0l},$$

$$R_{kl} = \left[a_l(\cos\varphi_l + \mu_k \sin\varphi_l) + ib_l(\sin\varphi_l - \mu_k \cos\varphi_l)\right]/2,$$

$$m_{kl} = \left[a_l(\cos\varphi_l + \mu_k \sin\varphi_l) - ib_l(\sin\varphi_l - \mu_k \cos\varphi_l)\right]/2R_{kl}.$$

Similarly, for the inclusion, the complex potentials are expanded in terms of Faber polynomials, resulting in a set of unknowns $a_{kp}^{(l)}$, which, together with a_{klp} are determined from the boundary conditions in the contact region. Once the complex potentials are determined, the bending moments and transverse forces at any point of the plate can be evaluated.

If the inclusion $S^{(l)}$ reduces to a straight elastic line (with the corresponding hole turning into a slit), the moment intensity factors (MIFs) $k_1^{\pm}$ (corresponding to the moments $M_y^{(l)}$ in the local coordinate system with its origin at point O_l (see Fig. 1)) and $k_2^{\pm}$ (for the moments $H_{xy}^{(l)}$ in the same coordinate system) can be evaluated, where the superscripts + and − denote the left and right ends of the inclusion, respectively. Following an analogy with the derivation of stress intensity factors (SIFs) for crack tips for plane elasticity problem [3], we obtain the following expressions.

$$k_1 = 2\operatorname{Re}\sum_{k=1}^{2}\left[p_k \sin^2\varphi_l + q_k \cos^2\varphi_l - r_k \sin 2\varphi_l\right]M_k ,$$

$$k_2 = 2\operatorname{Re}\sum_{k=1}^{2}\left[\frac{1}{2}\left(q_k - p_k\right)\sin 2\varphi_l + r_k \cos 2\varphi_l\right]M_k .$$

Numerical studies. Numerical investigations were performed for a plate made of isotropic material CAST-B (denoted as material M1) and glass-reinforced plastic with cross-ply winding (material M2). The deformation coefficients for these materials are given in Table 1. In the numerical calculations, the deformation coefficients for the inclusion material were selected as $a_{ij}^{(l)} = \lambda^{(l)} a_{ij}$, where $\lambda^{(l)}$ is the parameter of the relative stiffness of the inclusion $S^{(l)}$.

Table 1. Materials

Material	$a_{11}\cdot 10^{-4}, МПа^{-1}$	$a_{22}\cdot 10^{-4}, МПа^{-1}$	$a_{12}\cdot 10^{-4}, МПа^{-1}$	$a_{66}\cdot 10^{-4}, МПа^{-1}$
M1	72,100	72,100	-8,600	161,500
M2	10000	2,800	-0,770	27,000

Numerical investigations have shown that the values of the moments are significantly influenced by the aspect ratio of the ellipses. The calculations indicate that for $b_1/a_1 < 10^{-3}$ the inclusion can be treated as linear, and the moment intensity factors (MIFs) can be computed for it. As the ratio b_1/a_1 decreases, the absolute values of the moments M_s near the tips of the major semi-axis a_1 increase sharply, and for small values of b_1/a_1 the MIFs can be considered. The effect of the relative stiffness of the linear inclusion on the MIFs has also been established: for $\lambda^{(1)} < 10^{-3}$ the inclusion can be regarded as perfectly rigid, while for $\lambda^{(1)} > 10^{3}$ it behaves as perfectly compliant (a crack). When $10^{-3} < \lambda^{(1)} < 10^{3}$ the MIF values are very small and can be neglected.

Numerical studies have also been performed for a configuration involving two linear inclusions. For the isotropic material M1 Fig. 2 shows the variation of the ratio of the MIF k_1^{+} at the right tip of the left inclusion to the corresponding

MIF value for a plate with a single inclusion (k_1^+ / k_{10}^+). The results indicate that a decrease in the distance between the linear inclusions leads to a considerable increasing of the MIF values at the inner tips of the inclusions.

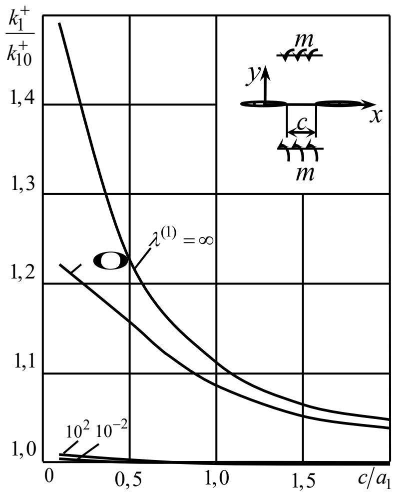

Fig. 2

Conclusions. Numerical results confirm that the geometry of elastic inclusions (their shape and spacing) as well as their stiffness relative to the surrounding material, significantly affect the distribution of bending moments in multiply connected plates. In cases of very stiff or very compliant inclusions, high stress concentrations arise near the inclusion tips. These effects are effectively characterized using moment intensity factors, which provide a convenient tool for evaluating the severity of stress state in such configurations.

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