TORSION OF A TRANSVERSELY ISOTROPIC ELASTIC-PLASTIC SOLID WITH A HEALED DISC-SHAPED CRACK

VITALII MELNYK

Karpenko Physico-Mechanical Institute of NAS of Ukraine

We develop a mathematical model for healing a disc-shaped crack in a transversely isotropic elastoplastic cylindrical body under torsion. The boundary-value problem is reduced to an integral equation for the angular displacement on the crack faces. For the case of a fully filled crack, a closed-form analytical solution is obtained. The strengthening effect is quantified in terms of the crack geometry and the mechanical properties of the cured injection material.

Keywords: crack healing, transversely isotropic cylinder, elastoplastic material, strength, anisotropy, torsion

Introduction. Injection repair is widely used to restore damaged concrete and reinforced-concrete members [1]. To keep such repaired structures reliable, we must estimate their remaining strength and service life. Some results are already available [2]. In this paper, we present a mathematical model of crack healing in a transtropic elastic–plastic cylinder under torsion.

Torsional strength of a circular cylinder with a crack. Let a long transversely isotropic cylinder contain in the plane of isotropy a disc-shaped crack of radius a (see figure 1). The cylinder is subjected to torsion by a moment M. The cylindrical coordinate system (r,θ,z) is chosen with the origin at the center of the crack so that the plane $z=0$ coincides with the plane of isotropy of the elastic properties. For such a loading scheme, the critical load M_C within the δ_C -concept is calculated from

$$\frac{8M_C\sqrt{n_3}}{3\pi^2R^4A_{44}}\left(2\sqrt{a_*^2-a^2}-I_2(a)\right)+\frac{3\tau_0\sqrt{n_3}}{\pi A_{44}}I_1(a)=\frac{\delta_{IIIc}}{a}, \tag{1}$$

where

$$I_1(r)=\int_\eta^R\frac{1}{t^2\sqrt{t^2-r^2}}\left(a\sqrt{t^2-a^2}+t^2\arccos\frac{a}{t}\right)dt,$$

$$I_2(r)=\int_\eta^R\frac{\sqrt{t^2-a^2}}{t^2\sqrt{t^2-r^2}}\left(2t^2+a^2\right)dt,\eta=\begin{cases}a, & r\le a,\\ r, & a<r\le a_*\end{cases},$$

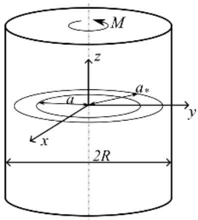

Fig. 1. Circular cylinder with a disk shaped crack.

τ_0 is the stress in the pre-failure zone near the crack. For an ideally elastic–plastic body, $\tau_0 = \tau_T$.

Integral Equations of the Problem on a Healed Crack. Assume that as a result of healing the crack by pressure injection of a fluid capable of polymerizing or crystallizing after some time, the crack faces in the region $0 \le r \le a_0$ $\left(a_0 \le a\right)$ are in mechanical contact with a thin elastic layer of the cured injection material. We will establish the strengthening effect resulting from healing the defect.

We use the principle of superposition and split the problem of torsion of a cylinder with a crack filled with another material into two auxiliary problems: torsion by the moment M of an intact body without a defect, and a body with a crack whose faces are subjected to tractions

$$\sigma_{z\theta}(r,0) = -f(r) = \begin{cases} -\frac{2Mr}{\pi R^4} + \frac{[u_\theta^*(r)]\mu_*}{2h(r)}, & 0 \le r \le a, \\ -\frac{2Mr}{\pi R^4} + \tau_0, & a < r \le a_*, \end{cases} \tag{2}$$

$$\sigma_{zr}(r,0) = \sigma_{zz}(r,0) = 0, 0 \le r \le a_*.$$

Here the thin elastic layer formed by the cured injection material is modeled according to a Winkler-type shear hypothesis; $2h$ is the layer thickness; μ_* is the shear modulus of the injection material after curing; u_θ^* is the displacement of the layer-surface points, which we take as the sum $u_\theta^*(r) = u_\theta(r) + u_\theta^0(r)$, $u_\theta^0(r) = 2Mrh(r)/\left(A_{44}\pi R^4\right)$; $H(r)$ is the Heaviside function; square brackets denote the jump of a function across the crack surface.

To solve the boundary-value problem (2), we assume R □ and consider the cylinder as an infinite space whose stresses satisfy the conditions (2) in the region $(0 \le r \le a, z = 0)$ and tend to those in a cylinder of radius R twisted by the moment M .

Satisfying these boundary conditions, we obtain for the unknown function $A(\xi)$ the pair of integral equations:

$$\int_0^a \xi A(\xi) J_1(r\xi) d\xi = \frac{f(r)}{\sqrt{\mu\tilde{\mu}}}. 0 \le r \le a,$$
$$\int_0^a A(\xi) J_1(r\xi) d\xi = 0, a \le r < \infty. \tag{3}$$

Here

$$f(r) = \frac{2Mr}{\pi R^4} - \frac{u_\theta \mu*}{h} H(a_0 - r) - \frac{2Mr\,\mu*}{\tilde{\mu}} H(a_0 - r). \tag{4}$$

Based on [3], the solution of this kind of dual integral equations has the form

$$A(\xi) = \sqrt{\frac{2\xi}{\pi \mu\tilde{\mu}}} \int^a \frac{J_{3/2}(\xi\eta)}{\sqrt{\eta}} d\eta \int_0^\eta \frac{r^2 f(r) dr}{\sqrt{\eta^2 - r^2}}, \tag{5}$$

where μ and $\tilde{\mu}$ are the shear moduli of the transversely isotropic material in the crack plane and in the transverse planes, respectively. Since the unknown crack-face displacement u_θ is determined by

$$u_\theta(r, 0) = \int_0^\infty A(\xi) J_1(r\xi), 0 \le r \le a, \tag{6}$$

then, using (5), we obtain the equation for determining these displacements:

$$u_\theta(r) = \frac{\sqrt{2r}}{\pi\sqrt{\mu\tilde{\mu}}} \int^a \frac{d\eta}{\sqrt{\eta^2 - r^2}} \int_0^\eta \frac{\rho^2 f(\rho) d\rho}{\sqrt{\eta^2 - \rho^2}}. \tag{7}$$

The integral equation (7) can be solved numerically. If we assume that the stress state in the injection layer depends only weakly on the pre-failure zone near the crack, we obtain an analytical solution in the form

$$u_\theta(r) = \frac{4r\sqrt{a_*^2 - r^2}\sqrt{n_3}}{3\pi A_{44}} \cdot \frac{2M\varepsilon}{\pi R^4} \cdot \frac{3\pi + 4\beta\varepsilon + 4\beta(1-\varepsilon)\sqrt{n_3}}{3\pi + 4\beta\varepsilon} +$$
$$+ \frac{rI_1(r)\sqrt{n_3}}{\pi A_{44}}\left(-\frac{2M_r}{\pi R^4} + \tau_0\right) - \frac{2rI_2(r)\sqrt{n_3}}{3\pi A_{44}}\left(-\frac{2M_r}{\pi R^4} + \varepsilon_0\right). \tag{8}$$

From the condition of smooth closure of the crack faces at the points $r = a*$, which is part of the δ_c-concept of fracture mechanics, we obtain the equation for determining the size of the pre-failure zone $a*$:

$$\frac{2\cdot 2ME_1}{\pi R^4}\frac{3\pi+4\beta\varepsilon+4\beta(1-\varepsilon)\sqrt{n_3}}{3\pi+4\beta\varepsilon}\left(2a_*^3-\sqrt{a_*^2-a^2}\left(a_*^2-a^2\right)\right)+$$
$$+3\left(-\frac{2M_r}{\pi R^4}+\tau_0\right)\left(a\sqrt{a_*^2-a^2}+a_*^2\arccos\frac{a}{a*}\right)=0. \tag{9}$$

The condition for the propagation of a longitudinal shear (Mode III) crack within the δ_c-model is

$$\left[u_\theta(a)\right]=\delta_{IIIc}. \tag{10}$$

Hence, taking into account (8), we obtain the equation for determining the critical moment M_c:

$$\frac{16M_c\varepsilon\sqrt{n_3}}{3\pi A_{44}\pi R^4}\cdot\frac{3\pi+4\beta\varepsilon+4\beta(1-\varepsilon)\sqrt{n_3}}{3\pi+4\beta\varepsilon}\sqrt{a_*^2-a^2}+$$
$$+\frac{2I_1(a)\sqrt{n_3}}{\pi A_{44}}\left(-\frac{2M_ca}{\pi R^4}+\tau_0\right)-\frac{4I_2(a)\sqrt{n_3}}{3\pi A_{44}}\left(-\frac{2M_ca}{\pi R^4}+\varepsilon_0\right)=\frac{\delta_{IIIc}}{a}. \tag{11}$$

Conclusion. We developed a mathematical model for torsion of a transversely isotropic elastic–plastic cylinder with a healed disc-shaped crack, reduced the boundary-value problem to dual integral equations for the crack-face angular displacement, and—for a fully filled crack—derived a closed-form solution within a Winkler-type interlayer model. From this framework we obtained explicit relations for the pre-failure zone size and the critical torsional moment, quantitatively demonstrating the strengthening effect of injection repair as a function of crack geometry and the elastic properties of the cured material

Acknowledgement. *This work was carried out as part of National Research Foundation of Ukraine project No. 2023.04/0132*

1. *Czarnecki L., Emmons P.H. Naprava i ochrona konstrukcji betonowych polski cement. – Krakow. – 2002. (in Polish)*
2. *Panasyuk V.V., Marukha V.I., Sylovanyuk V.P. Injection technologies for the repair of damaged concrete structures. – Switzerland: Springer Int. Publ. – 2013. – 230 p. https://doi.org/10.1007/978-94-007-7908-2*
3. *Sneddon I.H. Mixed boundary value problems in potential theory. – Amsterdam: North Holland Publishing Company. – 1986.*