HILBERT TRANSFORM OF BIPERIODICALLY NON-STATIONARY RANDOM SIGNALS

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An analysis of the covariance and spectral structure of the Hilbert transform of biperiodically non-stationary random processes, which are a model of signals with double rhythmicity, are analyzed here. The obtained relations join the cross-covariance and cross-spectral characteristics of the signal and its Hilbert transform with the characteristics of the signal itself.

Keywords: biperiodically non-stationary random process, Hilbert transform, covariance and spectral components, analytic signal

Introduction. In the analysis of signals of both natural and artificial origin, cases often occur when the stochastic repeatability of one period interacts with the repeatability of another one [1, 2]. In the case of vibration signals generated by mechanical systems, birhythmic variability is caused by diferent rotation speeds of elements of rotating units. A probabilistic model of double rhythmicity is biperiodically non-stationary random processes (BPNRP) [1, 3, 4]. The covariance-spectral structure of BPNRP is determined by jointly stationary processes that model the carrier harmonics, whose frequencies are linear combinations of two basic frequencies. The sum of the square of the signal and its Hilbert transform, i.e., the square of the modulus of the analytic signal, cannot be considered as the square of the envelope, since the stochastic properties of the Hilbert transform and the signal itself are the same [5, 6]. The newest obtained results have changed the principles of using the Hilbert transform in vibration diagnostics. Since stochastic variability with double rhythmicity is characteristic of vibrations of many defective mechanisms, the problem of establishing the characteristic features of the Hilbert transform of BPNRP that describe such vibrations is important.

Covariance and spectral properties of BPNRP. The mean function of a BPNRP $m_\xi(t) = E\xi(t)$ and its covariance function $b_\xi(t,u) = E\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t+u)$, $\overset{\circ}{\xi}(t) = \xi(t) - m_\xi(t)$, where E is the mathematical expectation operator, are biperiodic functions of time and can be represented by Fourier series:

$$m_\xi(t) = \sum_{k,l\in Z} m_{kl}^{(\xi)} e^{i\omega_{kl}t} =$$

$$= m_{00}^{(\xi)} + \sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\left(m_{kl}^{c}\cos\omega_{kl}t + m_{kl}^{s}\sin\omega_{kl}t\right) +$$

$$+\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\left(m_{k,-l}^{c}\cos\omega_{k,-l}t + m_{k,-l}^{s}\sin\omega_{k,-l}t\right), \tag{1}$$

$$b_{\xi}(t,u) = \sum_{k,l\in Z} B_{kl}^{(\xi)}(u)e^{i\omega_{kl}t} =$$

$$= B_{00}^{(\xi)}(u) + \sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\left[B_{kl}^{c}(u)\cos\omega_{kl}t + B_{kl}^{s}(u)\sin\omega_{kl}t\right] +$$

$$+\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\left[B_{k,-l}^{c}(u)\cos\omega_{k,-l}t + B_{k,-l}^{s}(u)\sin\omega_{k,-l}t\right]. \tag{2}$$

Here Z is the set of integers, $m_{kl}^{(\xi)} = \frac{1}{2}\left(m_{kl}^{c} - im_{kl}^{s}\right)$ and $B_{kl}^{(\xi)}(u) = \frac{1}{2}\left[B_{kl}^{c}(u) - iB_{kl}^{s}(u)\right]$, $m_{-k,-l}^{(\xi)} = \overline{m}_{kl}^{(\xi)}$, $B_{-k,-l}^{(\xi)}(u) = \overline{B}_{kl}^{(\xi)}(u)$, where "¯" – denotes conjugation, $\omega_{kl} = k\frac{2\pi}{P_1} + l\frac{2\pi}{P_2}$, P_1 and P_2 are periods. The process $\xi(t)$ can be represented in the form of a stochastic series:

$$\xi(t) = \sum_{k,l\in Z}\xi_{kl}(t)e^{iw_{kl}t}, \tag{3}$$

where $\xi_{kl}(t) = \frac{1}{2}\left[\xi_{kl}^{c}(t) - \xi_{kl}^{s}(t)\right]$, $\xi_{-k,-l}(t) = \overline{\xi}_{kl}(t)$, are jointly stationary random processes.

Hilbert transform. Consider that the random process $\xi(t)$ (3) has a zero^{th} constant component $m_{00} = 0$. Then, there exists the Hilbert transform

$$\eta(t) = H\{\xi(t)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\xi(\tau)}{t-\tau}d\tau, \tag{4}$$

and for its mathematical expectation, we have:

$$m_{\eta}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{m_{\xi}(t)}{t-\tau}d\tau.$$

After substituting into this formula series (1), we obtain:

$$m_{\eta}(t)=\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\left(m_{kl}^{c}\sin\omega_{kl}t-m_{kl}^{s}\cos\omega_{kl}t\right)+\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\left(m_{k,-l}^{c}\sin\omega_{k,-l}t-m_{k,-l}^{s}\cos\omega_{k,-l}t\right).$$

Here it is taken into account that the Hilbert transform shifts the phases of the harmonics of the mathematical expectation (1) by $-\frac{\pi}{2}$. The mathematical expectation of the analytic signal $\zeta(t)=\xi(t)+i\eta(t)$ then has the form:

$$m_{\zeta}(t)=2\left[\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}m_{kl}e^{i\omega_{kl}t}+\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}m_{k,-l}e^{i\omega_{k,-l}t}\right].$$

A BPNRP $\xi(t)$, whose covariance function is represented by the series (2), and its Hilbert transform (4) are jointly biperiodically non-stationary processes, and their auto- and cross-covariance components are related by the ratios:

$$B_{kl}^{(\xi\eta)}(u)=\int_{-\infty}^{\infty}h(u-\tau)B_{kl}^{(\xi)}(\tau)d\tau,$$

$$B_{kl}^{(\eta\xi)}(u)=-\int_{-\infty}^{\infty}h(u-\tau)B_{kl}^{(\eta)}(\tau)d\tau,$$

$$B_{kl}^{(\xi)}(u)=-\int_{-\infty}^{\infty}h(u-\tau)B_{kl}^{(\xi\eta)}(\tau)d\tau,$$

$$B_{kl}^{(\eta)}(u)=\int_{-\infty}^{\infty}h(u-\tau)B_{kl}^{(\eta\xi)}(\tau)d\tau,$$

where $h(\tau)=(\pi\tau)^{-1}$ is the impulse response of the Hilbert transform, which means that the covariance components $B_{kl}^{(\xi)}(u)$ and $B_{kl}^{(\xi\eta)}(u)$, as well as $B_{kl}^{(\eta\xi)}(u)$ and $B_{kl}^{(\eta)}(u)$, are Hilbert pairs.

The zero^{th} covariance component of a BPNRP signal and its Hilbert transform are equal, and their zero^{th} cross-covariance components differ only by sign, they are odd functions and are determined by the formula:

$$B_{00}^{(\xi\eta)}(u)=2\int_{0}^{\infty}f_{00}^{(\xi)}(\omega)\sin\omega u\,d\omega.$$

The cross-covariance function of a BPNRP signal and its Hilbert transform, as well as the autocovariance function of the latter, vary bi-periodically with time, and their Fourier coefficients are determined by the formulas:

$$B_{kl}^{(\eta)}(u)=\int_{-\infty}^{0} f_{kl}^{(\xi)}(\omega)e^{i\omega u}d\omega-\int_{0}^{\omega_{kl}} f_{kl}^{(\xi)}(\omega)e^{i\omega u}d\omega+\int_{\omega_{kl}}^{\infty} f_{kl}^{(\xi)}(\omega)e^{i\omega u}d\omega,$$

$$B_{kl}^{(\xi\eta)}(u)=i\int_{0}^{\infty}\left[f_{kl}^{(\xi)}(\omega+\omega_{kl})e^{-i\omega u}-f_{kl}^{(\xi)}(\omega)e^{i\omega u}\right]d\omega.$$

Conclusion. It was shown that the BPNRP signal and its Hilbert transform are jointly BPNRP, and their auto- and cross-covariance components form respective Hilbert pairs. It was proved that the zero^{th} covariance components of the BPNRP signal and its Hilbert transform are the same. The zero^{th} cross-covariance components differ only by sign and are odd functions. Conditions for the stationarity of the analytic signal were established. If these conditions are fulfilled, the non-zero covariance components of the signal and the Hilbert transform differ only by sign, the non-zero cross-covariance components are identical and are related with the signal's higher covariance components by the simple equation $B_{kl}^{(\xi\eta)}(u)=-iB_{kl}^{(\xi)}(u)$.

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